1. Proof of the Quotient Rule.

$$\left(\frac{f}{g}\right)'\Big|_{x=c} = \lim_{h\to 0} \frac{\frac{f(c+h)}{g(c+h)} - \frac{f(c)}{g(c)}}{h}$$

$$= \lim_{h\to 0} \frac{f(c+h)\cdot g(c) - g(c+h)\cdot f(c)}{h g(c)\cdot g(c+h)} \quad (\text{insert } f(c)g(c))$$

$$= \lim_{h\to 0} \frac{[f(c+h) - f(c)]\, g(c) + f(c)[g(c) - g(c+h)]}{h\cdot g(c)\cdot g(c+h)} \quad (*)$$

For $g(x)$ is differentiable at $x=c \Rightarrow g(x)$ is continous at $x=c$

so $\lim_{h\to 0} g(c+h) = g(c)$

And for $g(c)>0$, so $g(c+h)>0$ when $h\to 0$. this is the sign-preserving property of limit.

which means the denominator $h g(c)\cdot g(c+h)$ is well-defined, not equal to 0.

(Actually $g(c)<0$ is ok, only to make sure $g(c)\neq 0$).

then $(*)$:

$$(*) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}\cdot\frac{1}{g(c+h)} + \lim_{h\to 0}\frac{f(c)}{g(c)}\cdot\frac{g(c)-g(c+h)}{h}\cdot\frac{1}{g(c+h)}$$

$$= f'(c)\cdot\frac{1}{g(c)} - \frac{f(c)}{g(c)}\cdot g'(c)\cdot\frac{1}{g(c)}$$

$$= \frac{f'(c)\cdot g(c) - f(c)g'(c)}{g^2(c)}$$

2. "Rough proof" of chain-rule.

$$\frac{d f(g(x))}{dx}\Big|_{x=c} = \lim_{h\to 0}\frac{f(g(c+h)) - f(g(c))}{h}$$

$$= \lim_{h\to 0}\frac{f(g(c+h)) - f(g(c))}{g(c+h) - g(c)}\cdot\frac{g(c+h)-g(c)}{h} \quad (\text{regard } g(c)=y \quad g(c+h)=y_h.)$$

$$= \lim_{h\to 0}\frac{f(y_h) - f(y)}{y_h - y}\cdot\frac{g(c+h)-g(c)}{h} \quad (**).$$

f is differentiable at $y=g(c)$, due to the continuity of $g(x)$ at $x=c$

we know $g(c+h)=y_h \to y=g(c)$ when $h\to 0$.

$$\Rightarrow \frac{f(y_h)-f(y)}{y_h - y} \to f'(y) \text{ when } h\to 0.$$

then:

$(**) = f'(y)\cdot g'(c) = f'(g(c))\cdot g'(c)$ chain-rule formula.

Remark: The "rough" means we have problem for such proof.

we can insert $g(c+h)-g(c)$ only when $g(c+h)-g(c)\neq 0$, so this proof would fail when we consider the simplest case $g(x)=$ Constant, then $g(c+h)-g(c)=0$. Actually we only need there exists some points $\{h_n\}$ to make $g(c+h_n)-g(c)=0$, this proof fails.

3. First consider $\lim\limits_{x\to 0}\frac{f(x)}{g(x)}$ directly.

$\left|\frac{f(x)}{g(x)}\right| = \left|x\sin\frac{1}{x}\right| \leq |x|$, so apply sand-wich thm here we get:

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0} x\sin\frac{1}{x} = 0$$

On the other hand, use the L'Hospital Rule:

$$\lim_{x\to 0}\frac{f(x)}{g(x)} \overset{\text{"H"}}{=} \lim_{x\to 0}\frac{f'(x)}{g'(x)} = \lim_{x\to 0}\frac{2x\sin\frac{1}{x} - x^2\cdot\frac{1}{x^2}\cos\frac{1}{x}}{1} = \lim_{x\to 0}\left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right) \quad (3)$$

$x\sin\frac{1}{x}$ is well-done, but $\cos\frac{1}{x}$ doesn't have limit when $x\to 0$ for it is a oscillating function.

then (3) doesn't exist.

this example just shows a shortcoming of L'Hospital Rule. Only we know $\lim\limits_{x\to 0}\frac{f'(x)}{g'(x)}$ exists first can we make the conclusion $\lim\limits_{x\to 0}\frac{f(x)}{g(x)} = \lim\limits_{x\to 0}\frac{f'(x)}{g'(x)}$, and the non-existence of $\lim\limits_{x\to 0}\frac{f'(x)}{g'(x)}$ can't imply $\lim\limits_{x\to 0}\frac{f(x)}{g(x)}$ doesn't exist, just like this example shows.

4. A simple application of Taylor's thm. the error term:

$$E_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} \quad \xi\in(x,x_0) \text{ or } (x_0,x) \text{ here } x_0=0.$$

And $f^{(n+1)}(x) = a(a-1)\cdots(a-n)(1+x)^{a-(n+1)}$

$$\Rightarrow E_n = \frac{a(a-1)\cdots(a-n)(1+\xi)^{a-(n+1)}}{(n+1)!}x^{n+1}.$$